

ON FINITE GROUPS WHOSE DERIVED SUBGROUP HAS BOUNDED RANK

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Abstract

Let G be a finite group with derived subgroup of rank r . We prove that $|G : \mathbf{Z}_2(G)| \leq |G'|^{2r}$. Motivated by the results of I. M. Isaacs in [2] we show that if G is capable then $|G : \mathbf{Z}(G)| \leq |G'|^{4r}$. This answers a question of L. Pyber. We prove that if G is a capable p -group then the rank of $G/\mathbf{Z}(G)$ is bounded above in terms of the rank of G' .

1 Introduction

Let G be a finite group. We denote by $d(G)$ the minimal number of generators of G . The rank of G denotes the minimal number $r = \text{rk}(G)$ such that every subgroup of G can be generated by r elements. It is clear that $d(G) \leq \text{rk}(G) \leq \log_2 |G|$. The importance of the concept of rank is underlined by results due to A. Lubotzky and A. Mann (see [1]). There are various upper bounds in asymptotic group theory in which the logarithm of the order of certain subgroups occurs. Our goal is to replace this logarithm by the rank of the same subgroups. Related to a famous theorem of Schur, Wiegold [11] proved that if $|G : \mathbf{Z}(G)| = n$ then $|G'| \leq n^{\frac{1}{2} \log_2 n}$. R. Guralnick [5] gave a simple proof for this fact. Following a similar argument we obtain the following.

Theorem 1. *If G is a finite group with $\text{rk}(G/\mathbf{Z}(G)) = r$. Then*

$$|G'| \leq |G : \mathbf{Z}(G)|^{r+1}.$$

The extra-special groups show that there is no upper bound for the index of the centre in terms of the order of the derived subgroup. However, P. Hall (see [10] p.423) observed that $|G : \mathbf{Z}_2(G)|$ is bounded from above in terms of $|G'|$ (where $\mathbf{Z}_2(G)$ denotes the second member of the upper central series of G). We proved in [7] that

$$|G : \mathbf{Z}_2(G)| \leq |G'|^{2 \log_2 |G'|}.$$

L. Pyber [9] asked whether there is a constant c such that $|G : \mathbf{Z}_2(G)| \leq |G'|^{c \cdot \text{rk}(G')}$. In the present paper we prove that

Theorem 2. *If G is a finite group and $\text{rk}(G') = r$ then*

$$|G : \mathbf{Z}_2(G)| \leq |G'|^{2r}.$$

A group H is said to be capable if there exists some G such that $G/Z(G)$ is isomorphic to H . I. M. Isaacs [2] proved that if H is a capable group then $|H : Z(H)|$ is bounded above in terms of $|H'|$, or equivalently, if G is an arbitrary finite group then $|G : \mathbf{Z}_2(G)|$ is bounded above in terms of $|G' : G' \cap \mathbf{Z}(G)|$, but he has not given an explicit bound. We proved in [8] that if $|G' : G' \cap \mathbf{Z}(G)| = n$ then

$$|G : \mathbf{Z}_2(G)| \leq n^{2^{\log_2 n}}.$$

It was proved in [2] that if H' is cyclic in a capable group H and all the order 4 elements in H' are central then $|H : \mathbf{Z}(H)| \leq |H'|^2$. It was shown later in [8] that the condition on the order 4 elements can be omitted. Motivated by these results, L. Pyber [9] asked if there is a constant c such that

$$|H : \mathbf{Z}(H)| \leq n^{c \cdot \text{rk}(H')}$$

in every capable group H . We answer this question in the affirmative by showing that

Theorem 3. *If G is a finite group and $\text{rk}(G'/G' \cap \mathbf{Z}(G)) = r$ then*

$$|G : \mathbf{Z}_2(G)| \leq |G'/G' \cap \mathbf{Z}(G)|^{4r}.$$

Corollary 4. *If H is a finite capable group and $\text{rk}(H') = r$ then*

$$|H : \mathbf{Z}(H)| \leq |H'|^{4r}.$$

We obtain a better bound in the case when $\mathbf{Z}(G) = 1$. The crucial observation is as follows.

Theorem 5. *If G is a finite group with $\mathbf{Z}(G) = 1$ then $C_G(G') \leq G'$.*

As a consequence we obtain the following.

Theorem 6. *If G be a finite group with $\mathbf{Z}(G) = 1$ and $d = d(G')$ then $|G| \leq |G'|^{d+1}$.*

Finally we show the following bound for the rank of $G/\mathbf{Z}_2(G)$ in finite p -groups.

Theorem 7. *If G is a finite p -group and $\text{rk}(G'/G' \cap \mathbf{Z}(G)) = r$ then*

$$\text{rk}(G/\mathbf{Z}_2(G)) \leq \frac{1}{2}(13r^2 - r).$$

2 Proof of the results

We shall use the following well-known fact [1].

Lemma 8. *If $P = \langle a_1, a_2, \dots, a_d, Z(P) \rangle$ is a p -group then every element of P' is equal to a product of the form $[x_1, a_1][x_2, a_2] \dots [x_d, a_d]$ with $x_1, x_2, \dots, x_d \in P$.*

Proof of Theorem 1. Let P be a Sylow- p subgroup of G . It is well-known (see [3], p. 9) that $G' \cap P \cap Z(G) = P' \cap Z(G)$. It follows that

$$|G' \cap Z(G)|_p = |G' \cap P \cap Z(G)| = |P' \cap Z(G)| \leq |P'|.$$

By using Lemma 8 we obtain that

$$|P'| \leq \prod_{i=1}^d |\{[p, a_i] \mid p \in P\}| = \prod_{i=1}^d |P : C_P(a_i)| \leq |P : Z(P)|^d \leq$$

$$|P : Z(P)|^r \leq |P : P \cap Z(G)|^r = |G : Z(G)|_p^r.$$

Consequently,

$$|G'|_p \leq |G : Z(G)|_p |G' \cap Z(G)|_p \leq |G : Z(G)|_p^{r+1}.$$

Since this inequality holds for all prime divisors of $|G|$ the proof is complete. \square

Lemma 9. $Z_2(G) \leq C_G(G')$ and $[C_G(G'), C_G(G')] \leq Z(G)$. In particular $C_G(G')$ is nilpotent of class 2 and every Sylow- p subgroup of $C_G(G')$ is normal in G .

Proof. Using the Three Subgroup Lemma we obtain that

$$[G, G, Z_2(G)] \leq [Z_2(G), G, G] = [G, Z_2(G), G] = 1$$

and

$$[C_G(G'), C_G(G'), G] \leq [C_G(G'), G, C_G(G')] = [G, C_G(G'), C_G(G')] = 1.$$

Let P be a Sylow- p subgroup of $C_G(G')$. Since $C_G(G') \triangleleft G$ and P is characteristic in $C_G(G')$ we have that $P \triangleleft G$. \square

Lemma 10. Let $H, K \leq G$ two subgroups such that H can be generated by d elements and K is normal in G . Then $|K : C_K(H)| \leq |G' \cap K|^d$.

Proof. Let x be an arbitrary element of G . It is easy to see that $|K : C_K(x)| = |\{[x, k] \mid k \in K\}|$. Using that K is normal in G we have that $\{[x, k] \mid k \in K\} \subset K \cap G'$. It follows that $|K : C_K(x)| \leq |G' \cap K|$.

Let x_1, x_2, \dots, x_d be a generating system of H .

$$|K : C_K(H)| \leq \prod_{i=1}^d |K : C_K(x_i)| \leq |G' \cap K|^d.$$

\square

Corollary 11. If G is a finite group and $d(G') = d$ then $|G : C_G(G')| \leq |G'|^d$.

Proof. We apply Lemma 10 for $H = G'$ and $K = G$. \square

Proof of Theorem 5. Let P be a Sylow p -subgroup of $C_G(G')$. Using Lemma 9 we have that $P \triangleleft G$. It follows that the Abelian group G/G' acts on P under conjugation. Let H be the Sylow p -subgroup of G/G' and Q be its complement. Applying Fitting's Lemma ([4] p. 180) we obtain that $P = [P, Q] \times C_P(Q)$. Assume that $|C_P(Q)| > 1$. Since the p -group H acts on the p -group $C_P(Q)$ we have that $L = C_P(H) \cap C_P(Q)$ is nontrivial therefore L is central in G , which is a contradiction. Hence $P = [P, Q] \leq G'$. This holds for each Sylow p -subgroup of $C_G(G')$ which completes the proof. \square

Proof of Theorem 6. The result follows immediately from Corollary 11 and Theorem 5. \square

Lemma 12. *Let A be a finite Abelian p -group of rank r . Let \mathcal{S} be a collection of subgroups of A such that $\bigcap \mathcal{S} = \{1\}$. Then there exists a subset \mathcal{R} of \mathcal{S} such that $|\mathcal{R}| \leq r$ and $\bigcap \mathcal{R} = \{1\}$.*

Proof. Let $S = \text{Soc}(A)$ be the subgroup generated by the elements of order p . Since S is an elementary Abelian p -group of rank r we can construct a descending chain $S = S_0 > S_1 > \dots > S_l = 1$ where $l \leq r$ and

$$S_i = S_{i-1} \cap H_i \text{ for some } H_i \in \mathcal{S} \quad (1 \leq i \leq l)$$

Let $\mathcal{R} = \{H_1, H_2, \dots, H_l\}$. We have that $|\mathcal{R}| = l \leq r$ and that the intersection of $\bigcap \mathcal{R}$ and S is trivial. Since $\bigcap \mathcal{R}$ does not contain any element of order p it follows that $\bigcap \mathcal{R} = \{1\}$. \square

Lemma 13. *Let G be a finite group with $Z = G' \cap \mathbf{Z}(G)$ and $\text{rk}(G'/Z) = r$. Then $|C_G(G') : \mathbf{Z}_2(G)| \leq |G' : Z|^r$.*

Proof. Let p be a prime divisor of $|C_G(G')|$ and let P be a Sylow- p subgroup of $C_G(G')$. Using Lemma 9, P is normal in G . It is clear that $P \cap G'$ is an Abelian group. Now

$$\bigcap_{x \in G} C_{P \cap G'}(x) = P \cap Z.$$

Applying Lemma 12 with

$$A = P \cap G' / P \cap Z, \quad \mathcal{S} = \{C_{P \cap G'}(x) / P \cap Z \mid x \in G\}$$

we obtain that there exist elements x_1, x_2, \dots, x_l with $l \leq \text{rk}(P \cap G' / P \cap Z) \leq r$ such that

$$\bigcap_{1 \leq i \leq l} C_{P \cap G'}(x_i) = P \cap Z.$$

Let $T = \langle x_1, x_2, \dots, x_l \rangle$ and $M/Z = C_{G/Z}(TZ/Z)$. Now we apply the Three Subgroup Lemma for $P \cap M$, T and G . Since $P \cap M \leq C_G(G')$ we have that $[G, T, M \cap P] = 1$. Using that

$$[M \cap P, T] \leq [M, T] \leq Z$$

we obtain that $[M \cap P, T, G] = 1$. It follows that $[M \cap P, G, T] = 1$ and so

$$[M \cap P, G] \leq C_G(T) \cap P \cap G' \leq Z.$$

This implies that $M \cap P \leq \mathbf{Z}_2(G) \cap P$.

Let us denote \hat{G}' , \hat{T} , \hat{M} and \hat{P} the images of G' , T , M and P in the factor group G/Z . Applying the second statement of Lemma 10 for $\hat{K} = \hat{P}$ and $\hat{H} = \hat{T}$, and using that $Z \leq M$, one gets that

$$\begin{aligned} |P : P \cap \mathbf{Z}_2(G)| &\leq |P : P \cap M| = |PM : M| = |\hat{P}\hat{M} : \hat{M}| = \\ &= |\hat{P} : \hat{P} \cap \hat{M}| \leq |\hat{G}' : \hat{P} \cap \hat{G}'|^r \leq (n_p)^r \end{aligned}$$

where n_p denotes the p -part of $|G' : Z|$.

Let P_1, P_2, \dots, P_t be the unique Sylow subgroups of $C_G(G')$ corresponding to the prime divisors p_1, p_2, \dots, p_t of $|C_G(G')|$. By embedding the Sylow subgroups of $\mathbf{Z}_2(G)$ into Sylow- p subgroups of $C_G(G')$ we obtain that

$$|C_G(G') : \mathbf{Z}_2(G)| = \prod_{1 \leq i \leq t} |P_i : P_i \cap \mathbf{Z}_2(G)| \leq \prod_{1 \leq i \leq t} n_{p_i}^r = |G' : Z|^r.$$

□

Proof of Theorem 2. Since

$$|G : \mathbf{Z}_2(G)| = |G : C_G(G')| |C_G(G') : \mathbf{Z}_2(G)|$$

and

$$rk(G'/Z) \leq rk(G') = r$$

Corollary 11 and Lemma 13 complete the proof. □

Lemma 14. *Let G be a finite group, let $D = \{g \in G \mid [g, G'] \subseteq \mathbf{Z}(G)\}$ and let P be a Sylow- p subgroup of D . Then $G'/C_{G'}(P)$ is a p -group.*

Proof. Since $D/\mathbf{Z}(G)$ is the centralizer of the commutator subgroup of $G/\mathbf{Z}(G)$ Lemma 9 yields that $D/\mathbf{Z}(G)$ is nilpotent therefore D is nilpotent. It follows that P is normal in G . Let $q \neq p$ be a prime divisor of G' and let Q be a Sylow- q subgroup of G' . Applying Fitting's Lemma ([4] p. 180) for the action of Q on P we get that $P = [Q, P]C_P(Q)$. Since

$$[Q, P] \leq [G', D] \cap P \leq \mathbf{Z}(G) \cap P$$

we obtain that $P = C_P(Q)$ thus $Q \leq C_{G'}(P)$. Now we have that $C_{G'}(P)$ is a normal subgroup of G' which contains all Sylow- q subgroups of G' for all $q \neq p$ primes. □

We will make use of the following well known fact.

Lemma 15. *If F is a p -group with $d(F) = d$ then any generating set of F contains a subset of size d which is a generating set as well.*

Lemma 16. *Let G be a finite group, $D = \{g \in G \mid [g, G'] \subseteq \mathbf{Z}(G)\}$, $Z = G' \cap \mathbf{Z}(G)$ and $rk(G'/Z) = r$. Then $|D : C_G(G')| \leq |G' : Z|^r$.*

Proof. Let p be a prime divisor of $|D|$ and let P be the Sylow- p subgroup of D . According to Lemma 14, $G'/C_{G'}(P)$ is a p -group. Since G' is generated by the commutator words $[x, y]$ ($x, y \in G$) the factor $G'/C_{G'}(P)$ is generated by their images under the natural homomorphism. Using that $G'/C_{G'}(P)$ is a p -group and $rk(G'/C_{G'}(P)) \leq rk(G'/Z) = r$, Lemma 15 guarantees that there exist elements $x_1, y_1, x_2, y_2, \dots, x_r, y_r \in G$ such that

$$\langle [x_1, y_1], [x_2, y_2], \dots, [x_r, y_r], C_{G'}(P) \rangle = G'.$$

Let $T = \langle x_1, y_1, x_2, y_2, \dots, x_r, y_r \rangle$ and let $M/Z = C_{G/Z}(TZ/Z)$. We apply the Three Subgroup Lemma for T, T and M we get that

$$[T, T, M] \subseteq [T, M, T] = [M, T, T] = 1$$

which means that $M \leq C_G(T')$. Since $G' = T'C_{G'}(P)$ we have that $C_G(G') = C_G(T') \cap C_G(C_{G'}(P))$. It is clear that $P \leq C_G(C_{G'}(P))$ and so

$$C_G(G') \cap P = C_G(T') \cap P \geq M \cap P.$$

Let us denote \hat{G}' , \hat{T} , \hat{M} and \hat{P} the images of G' , T , M and P in the factor group G/Z . Applying the second statement of Lemma 10 for $\hat{K} = \hat{P}$ and $\hat{H} = \hat{T}$, and using that $Z \leq M$, one gets that

$$\begin{aligned} |P : P \cap C_G(G')| &\leq |P : P \cap M| = |PM : M| = |\hat{P}\hat{M} : \hat{M}| = \\ &= |\hat{P} : \hat{P} \cap \hat{M}| \leq |\hat{G}' : \hat{P} \cap \hat{G}'|^{2r} \leq (n_p)^{2r}. \end{aligned}$$

where n_p denotes the p -part of $|G' : Z|$.

Let P_1, P_2, \dots, P_t be the unique Sylow subgroups of D corresponding to the prime divisors p_1, p_2, \dots, p_t of $|D|$. By embedding the Sylow subgroups of $C_G(G')$ into Sylow- p subgroups of D we obtain that

$$|D : C_G(G')| = \prod_{1 \leq i \leq t} |P_i : P_i \cap C_G(G')| \leq \prod_{1 \leq i \leq t} n_{p_i}^{2r} = |G' : Z|^{2r}.$$

□

Proof of Theorem 3. Let D be as in Lemma 14. Since

$$|G : D| = |G/Z : C_{G/Z}(G'Z/Z)|$$

it follows by Lemma 11, Lemma 16 and Corollary 13 that

$$|G : \mathbf{Z}_2(G)| = |G : D| |D : C_G(G')| |C_G(G') : \mathbf{Z}_2(G)| \leq |G'/Z|^{4r}.$$

□

Lemma 17. *Let G be a finite group and let x be an element of G . Then the map $a \rightarrow [a, x]$ is a homomorphism from $C_G(G')$ to G' .*

Proof. Assume that $a, b \in C_G(G')$. Then

$$[ab, x] = [a, x]^b [b, x] = [a, x][b, x].$$

□

Lemma 18. *Let G be a finite p -group with $Z = G' \cap \mathbf{Z}(G)$ and $\text{rk}(G'/Z) = r$. Then $\text{rk}(C_G(G') : \mathbf{Z}_2(G)) \leq r^2$.*

Proof. We import the notation and calculations from the proof of Lemma 13. Since G is a p -group we have that $P = C_G(G')$, $P \cap G' = \mathbf{Z}(G')$, $P \cap \mathbf{Z}_2(G) = \mathbf{Z}_2(G)$ and $P \cap Z = Z$. Recall that there are elements x_1, x_2, \dots, x_l with $l \leq \text{rk}(C_G(G') \cap G'/Z) \leq r$ such that if $T = \langle x_1, x_2, \dots, x_l \rangle$ and $M/Z = C_{G/Z}(TZ/Z)$ then $M \cap P \leq \mathbf{Z}_2(G) \cap P$ or equivalently

$$M \cap C_G(G') \leq \mathbf{Z}_2(G).$$

For each $1 \leq i \leq l$ let $f_i : C_G(G') \rightarrow G'/Z$ be the map given by

$$f_i(x) = [x, x_i]Z.$$

Lemma 17 implies that f_i is a homomorphism for all $1 \leq i \leq l$ and we have that

$$\bigcap_{1 \leq i \leq l} \ker(f_i) = M \cap C_G(G') \leq \mathbf{Z}_2(G).$$

It follows that the map $x \rightarrow (f_1(x), f_2(x), \dots, f_l(x))$ embeds $C_G(G')/(M \cap C_G(G'))$ into the l -th direct power of G'/Z . Consequently

$$\mathrm{rk}(C_G(G')/\mathbf{Z}_2(G)) \leq \mathrm{rk}(C_G(G')/(M \cap C_G(G'))) \leq \mathrm{rk}((G'/Z)^l) \leq lr \leq r^2.$$

□

Lemma 19. *Let G be a finite p -group, $D = \{g \in G \mid [g, G'] \subseteq \mathbf{Z}(G)\}$, $Z = G' \cap \mathbf{Z}(G)$ and $\mathrm{rk}(G'/Z) = r$. Then $\mathrm{rk}(D/C_G(G')) \leq 2r^2$.*

Proof. We borrow the notation and calculations from Lemma 16. Since G is a p -group we have that $D = P$. Recall that we constructed elements

$$x_1, y_1, s_2, y_2, \dots, x_r, y_r \in G$$

where $r = \mathrm{rk}(G'/Z)$ such that if

$$T = \langle x_1, y_1, x_2, y_2, \dots, x_r, y_r \rangle$$

and

$$M/Z = C_{G/Z}(TZ/Z)$$

then

$$C_G(G') = C_G(G') \cap D \geq M \cap D. \quad (1)$$

Note that $D/\mathbf{Z}(G)$ is the centralizer of the commutator subgroup in $G/\mathbf{Z}(G)$. It follows by Lemma 17 that for any fixed element $x \in G$ the map $f_x(a) = [x, a]Z$ is a homomorphism on D . That the map

$$a \rightarrow (f_{x_1}(a), f_{y_1}(a), f_{x_2}(a), f_{y_2}(a), \dots, f_{x_r}(a), f_{y_r}(a))$$

embeds $D/(M \cap D)$ into the $2r$ -th direct power of G'/Z . Using (1) we obtain that $\mathrm{rk}(D/C_G(G')) \geq 2r^2$. □

We will need the following Theorem from [6] (see Proposition 16, p.363).

Theorem 20. *Let P be a finite p -group of rank r and Q a p -subgroup of $\mathrm{Aut}(P)$. Then the rank of Q is at most $\frac{1}{2}(5r^2 - r)$ if p is odd, at most $\frac{1}{2}(7r^2 - r)$ if $p = 2$.*

Proof of Theorem 4. Let $D = \{g \in G \mid [g, G'] \subseteq \mathbf{Z}(G)\}$ and $z = G' \cap \mathbf{Z}(G)$ as usual. We have that

$$\mathrm{rk}(G/\mathbf{Z}_2(G)) \leq \mathrm{rk}(G/D)\mathrm{rk}(D/C_G(G'))\mathrm{rk}(C_G(G')/\mathbf{Z}_2(G))$$

Considering the action of $G/\mathbf{Z}(G)$ on G'/Z we get that the group G/D is embedded into the automorphism group of G'/Z . According to Theorem 20 we have that $\mathrm{rk}(G/D) \leq \frac{1}{2}(7r^2 - r)$ where $r = \mathrm{rk}(G'/Z)$. Now together with Lemma 18 and Lemma 19 we get that

$$\mathrm{rk}(G/\mathbf{Z}_2(G)) \leq \frac{1}{2}(13r^2 - r).$$

□

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